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INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 15135–15145 doi:10.1088/0305-4470/39/49/005

On linearization of third-order ordinary differential equations

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Received 9 August 2006, in final form 27 October 2006
Published 21 November 2006
Online at stacks.iop.org/JPhysA/39/15135

Abstract
A new algorithm for linearization of a third-order ordinary differential equation is presented. The algorithm consists of composition of two operations: reducing order of an ordinary differential equation and using the Lie linearization test for the obtained second-order ordinary differential equation. The application of the algorithm to several ordinary differential equations is given.

PACS numbers: 02.10.Ud, 02.30.Hq

1. Introduction

Many methods of solving differential equations use a change of variables that transforms a given differential equation into another equation with known properties. Since the class of linear equations is considered to be the simplest class of equations, there is the problem of transforming a given differential equation into a linear equation. This problem, which is called a linearization problem, is a particular case of an equivalence problem. The equivalence problem can be formulated as follows. Let a set of invertible transformations be given. One can introduce the equivalence property according to these transformations: two differential equations are equivalent if there is a transformation of the given set which transforms one equation into another. The equivalence property separates all differential equations into classes of equivalent equations. Assume that there are two equations. The equivalence problem is: do these two equations belong to the same class. This problem involves a number of related problems such as defining a class of transformations, finding invariants of these transformations, obtaining the equivalence criteria and constructing the transformation.

For the linearization problem one studies the classes of equations equivalent to linear equations.

1.1. Second-order equation: the Lie linearization test

The first linearization problem for ordinary differential equations was solved by Lie [1]. He found the general form of all ordinary differential equations of second order that can be reduced to a linear equation by changing the independent and dependent variables. He showed that any linearizable second-order equation should be at most cubic in the first-order derivative and provided a linearization test in terms of its coefficients. The linearization criterion is written through relative invariants of the equivalence group. Tresse [2] treated the equivalence problem for second-order ordinary differential equations in terms of relative invariants of the equivalence group of point transformations. In [3], an infinitesimal technique for obtaining relative invariants was applied to the linearization problem.

Since the Lie test is the main tool of the algorithm developing in the manuscript, let us recall it in detail.

The Laguerre–Forsyth canonical form of a second-order linear equation with the independent variable t and the dependent variable u is

$$u'' = 0. \tag{1}$$

Lie [1] showed that any second-order ordinary differential equation $y'' = F(x, y, y')$ obtained from a linear equation by a change of the independent and dependent variables,

$$t = \varphi(x, y), \qquad u = \psi(x, y), \tag{2}$$

is cubic in the first derivative[1]:

$$y'' + a(x, y)y'^3 + b(x, y)y'^2 + c(x, y)y' + d(x, y) = 0. \tag{3}$$

Moreover, a second-order ordinary differential equation is linearizable if and only if it has the form (3) with the coefficients satisfying the conditions

$$\begin{aligned}&3a_{xx} - 2b_{xy} + c_{yy} - 3a_x c + 3a_y d + 2b_x b - 3c_x a - c_y b + 6d_y a = 0,\\ &b_{xx} - 2c_{xy} + 3d_{yy} - 6a_x d + b_x c + 3b_y d - 2c_y c - 3d_x a + 3d_y b = 0.\end{aligned} \tag{4}$$

Linearizing transformations are found by solving involutive systems of partial differential equations. These systems depend on the coefficient a.

If $a = 0$, then $\varphi = \varphi(x)$ and the involutive system is

$$\begin{aligned}&\psi_{yy} = \psi_y b, \qquad 2\psi_{xy} = \varphi_x^{-1}\psi_y\varphi_{xx} + c, \qquad \psi_{xx} = \varphi_x^{-1}\psi_x\varphi_{xx} + \psi_y d,\\ &2\varphi_x\varphi_{xxx} - 3\varphi_{xx}^2 - \varphi_x^2(4(d_y + bd) - (2c_x + c^2)) = 0.\end{aligned} \tag{5}$$

If $a \neq 0$, then $\varphi_y \neq 0$ and the functions $\varphi(x, y)$ and $\psi(x, y)$ satisfy the involutive system of partial differential equations

$$\begin{aligned}&\varphi_y\psi_{yy} = (\varphi_{yy}\psi_y + a\Delta), \qquad 2\varphi_y^2\psi_{xy} = 2\varphi_{xy}\varphi_y\psi_y - \varphi_{yy}\Delta - (a\varphi_x - b\varphi_y)\Delta,\\ &\varphi_y^2\psi_{xx} = 2\varphi_{xy}\varphi_y\psi_x - \varphi_x\varphi_{yy}\psi_x - \varphi_x^2\psi_x a + \varphi_x\varphi_y\psi_x b + \varphi_y^2(\psi_y d - \psi_x c),\\ &\varphi_y^2\varphi_{xx} = 2\varphi_{xy}\varphi_x\varphi_y - \varphi_x^2\varphi_{yy} - \varphi_x^3 a + \varphi_x^2\varphi_y b - \varphi_x\varphi_y^2 c + \varphi_y^3 d,\\ &2\varphi_y\varphi_{yyy} = 3\big(\varphi_{yy}^2 - 2\varphi_{xy}\varphi_y a + 2\varphi_x\varphi_{yy} a + \varphi_x^2 a^2\big) - 2\varphi_x\varphi_y(a_y + ab)\\ &\qquad + \varphi_y^2(2b_y - 4a_x + 4ac - b^2),\\ &6\varphi_y^2\varphi_{xyy} = 3\big(4\varphi_{xy}\varphi_{yy}\varphi_y - \varphi_x\varphi_{yy}^2 + 2\varphi_x\varphi_{yy}\varphi_y b - 2\varphi_{xy}\varphi_y^2 b\big) + 3\varphi_x^3 a^2\\ &\qquad + 3\varphi_x\varphi_y^2(-2a_x + 2ac - b^2) + 2\varphi_y^3(-b_x + 2c_y + 3ad),\end{aligned} \tag{6}$$

where $\Delta = \varphi_x\psi_y - \varphi_y\psi_x \neq 0$ is the Jacobian of the change of variables.

A different approach for tackling the equivalence problem of second-order ordinary differential equations was developed by Cartan [5]. The idea of his approach was to associate with every differential equation a uniquely defined geometric structure of a certain form.

[1] Details can be found in [4].

1.2. Linearization of third-order ordinary differential equations

The Cartan approach was further applied by Chern [6] to third-order ordinary differential equations. Note that according to the Laguerre–Forsyth theorem a canonical form of a linear third-order ordinary differential equation is

$$u''' + \alpha u = 0, \tag{7}$$

where $\alpha = \alpha(t)$. Chern studied the linearization problem for a third-order ordinary differential equation which is equivalent to (7) with constant α.

The linearization problem of a third-order ordinary differential equation with respect to point transformations was studied in [7]. Complete criteria for linearization were obtained in [8, 9].

Lie also noted that all second-order ordinary differential equations can be transformed to each other by means of contact transformations, and that this is not so for third-order ordinary differential equations. Hence, the linearization problem by a contact transformation also becomes interesting for a third-order ordinary differential equation. This problem was studied in [10–13]. The solutions of the linearization problem were given in [14] and [9][2]. In an explicit form the criteria for linearization are presented in [9].

The linearization problem for a third-order ordinary differential equation was also investigated with respect to a generalized Sundman transformation[3] [18, 19]:

$$u(t) = F(x, y), \qquad \mathrm{d}t = G(x, y)\,\mathrm{d}x.$$

Criteria for a third-order ordinary differential equation to be equivalent to the linear equation

$$u''' = 0$$

with respect to a Sundman transformation were presented in [19].

It is known that all discussed above methods (point, contact or generalized Sundman transformations) are complementary: there are equations that can only be linearized by one of these methods.

Recently the extension of the generalized Sundman transformation (GLT)

$$u(t) = F(x, y), \qquad \mathrm{d}t = G(x, y, y')\mathrm{d}x \tag{8}$$

were presented in [17, 20]. An example[4] of an equation which can be linearized by a transformation of the form (8) is given in [20]. It is worth noting that any second-order equation $y'' = f(x, y, y')$ can be transformed by a transformation (8) to the free particle equation. Hence, similar to contact transformations, only special transformations can be applied for the classification of second-order equations. In [17], the authors applied a particular class of transformations (8), where the function $G(x, y, y')$ is linear with respect to y'.

1.3. Reducing order of an ordinary differential equation

A linearizing transformation allows reducing an equation to a quadrature. The reduction of the order of a nonlinear ordinary differential equation is another procedure in the reduction of this equation to quadratures. The reduction in order can be done by finding an integrating factor or by using Lie group symmetries.

[2] The authors of [9] discovered [14] after obtaining the results of [9]. Because the criteria for linearization of a third-order ordinary differential equation by a contact transformation given in [14] were in implicit form, it was decided to include the obtained results in [9].

[3] Generalized Sundman transformation was also applied to second-order ordinary differential equations in [15–17].

[4] Other examples studied in [20] can be linearized by either point or contact or generalized Sundman transformations.

Although in principle it is always possible to determine whether a given ordinary differential equation is exact (a total derivative), there is no known method which is always successful in making an arbitrary equation exact. One can find a discussion on obtaining an integrating factor for nth-order ordinary differential equation in [21–24] and references therein. A major problem in finding an integrating factor consists of solving a system of determining equations. Most authors solve this problem by exploring possible ansatzes. In [20], integrating factors were applied for a linearization problem.

If an ordinary differential equation admits a Lie group of transformations, then by using canonical variables one can transform this equation to an autonomous ordinary differential equation. It is well known that the order of an autonomous ordinary differential equation can be reduced. For example, a third-order ordinary differential equation

$$y''' = F(y, y', y'')$$

by denoting $y' = f(y)$ is reduced to the second-order ordinary differential equation:

$$f^2 f'' + (f')^2 = F(y, f, ff').$$

If a solution of the last equation is found, then a solution of the original equation is obtained in a quadrature.

1.4. Content of the manuscript

The manuscript is devoted to developing a new criterion for the linearization of a third-order ordinary differential equation. The algorithm consists of the composition of two operations: reducing order of an ordinary differential equation and using the linearization test for the obtained equation. The algorithm can also be applied to higher order ordinary differential equations. Linearization of some fourth-order ordinary differential equations is also given in the manuscript. Thus, one can study the hierarchy of linearization methods.

Despite the simplicity of the presented algorithm, it allows linearizing many equations that cannot be linearized by point, contact or generalized Sundman transformations. Examples show that the novel algorithm is complementary to these linearization methods. Most examples presented in the manuscript cannot be linearized by other methods which described above except the novel algorithm. It is worth mentioning that among the examples one can find such well-known equations as describing travelling waves of Korteveg–de Vries, Camassa–Holm, Kadomtsev–Petviashvili and some other equations. Note also that after obtaining the linearizing transformation the general solution of the original equation is obtained in quadrature.

As far as the author's knowledge is concerned no such single algorithm has been applied in the literature.

The manuscript is organized as follows.

In section 2, criteria for the linearization of an autonomous third-order ordinary differential equation are presented. It is shown that the well-known third-order ordinary differential equations[5] ((8.32) and (8.33) in [25], section 8.3.3), which are linearizable by contact transformations, can also be linearized by the novel algorithm. The new linearization is much simpler than the linearization by contact transformations.

Section 3 considers stationary solutions of equations of fluid with internal inertia. The idea of a new way for linearizing ordinary differential equations was obtained during the study of stationary solutions of the equations [26].

[5] Discussion on the linearization of these equations one can find in [9, 25].

Section 4 considers travelling waves of some partial differential equations: the Korteveg–de Vries equation, the equation which includes such as Fornberg-Whitham equation, the Rosenau–Hyman equation, the Fuchssteiner–Fokas–Camassa–Holm equation, the Benjamin–Bona–Mahoney equation. Linearization of some fourth-order equations is also considered in this section (the Boussinesq equation, one class of fourth-order partial differential equations which was studied in [27]).

2. New algorithm

If the order of a third-order ordinary differential equation is reduced, then one can apply the Lie test for linearization. If the reduced second-order equation satisfies the Lie linearization test, then after finding a linearizing change of the independent and dependent variables, the general solution of the original equation is obtained in quadrature. Study of many cases shows that this method of linearization is very effective. Moreover, it linearizes many equations which are linearized by too sophisticated methods. This manuscript is devoted to prove this statement.

Since the linearization test for a third-order ordinary differential equation is also obtained [8, 9], one can use a similar procedure for a fourth-order ordinary differential equation. For the sake of simplicity the method is described for a third-order ordinary differential equation.

2.1. Third-order ODE equivalent to an autonomous ODE

One of the first problems of applying the algorithm is the problem of reducing the order of an equation. If a third-order ordinary differential equation admits a Lie group, then using canonical variables (x, v) one can map this equation into an autonomous equation

$$v''' = F(v, v', v''). \tag{9}$$

Assuming $v' = y(v)$, the last equation is transformed to the equation

$$y^2 y'' + y(y')^2 - F(v, y, yy') = 0. \tag{10}$$

The Lie test gives necessary and sufficient conditions for equation (10) to be linearizable. These conditions are

$$F(v, v', v'') = v''^3 a_3(v, v') + v''^2 a_2(v, v') + v'' a_1(v, v') + a_0(v, v')$$

and the coefficients $a_i = a_i(x, y)$, $(i = 0, 1, 2, 3)$ have to satisfy the conditions

$$\begin{aligned}
&a_{2xx} y^4 - y^3(2a_{1xy} + a_{2x}a_1 - 3a_{0x}a_3 - 6a_{3x}a_0)\\
&+y^2(3a_{0yy} + 2a_{1x} + 2a_{1y}a_1 - 3a_{2y}a_0 - 3a_{0y}a_2) - y\big(9a_{0y} - 6a_0a_2 + 2a_1^2\big) + 9a_0 = 0\\
&3a_{3xx} y^4 - y^3(2a_{2xy} + 2a_{2x}a_2 - 3a_{3x}a_1 - 3a_{1x}a_3)\\
&+y^2(a_{1yy} + a_{1y}a_2 + 2a_{2x} - 3a_{3y}a_0 - 6a_{0y}a_3) - y(3a_{1y} - 9a_0a_3 + a_1a_2) + 3a_1 = 0.
\end{aligned} \tag{11}$$

Thus the difficulties in the application of the novel method are similar to the difficulties in the Lie test (4).

2.2. Computer applications

The linearization criteria (point, contact, generalized Sundman transformations or the novel algorithm) look very massy, but for a given equation their checking on a computer takes only a few seconds. All criteria of linearization are assembled in one computer program using Reduce [28]. The author checked many equations for linearization. The result of these calculations confirms that all discussed above methods (point, contact or generalized Sundman transformations) and the novel algorithm are complementary: there are equations that can only be linearized by one of these methods.

2.3. Some examples

It is known [29] (see also [25], v.3, section 8.3.3)) that the equations

$$(C) v''' = \frac{3v'v''^2}{1+v'^2} \qquad (H) v''' = \frac{3v''^2}{2v'} \tag{12}$$

are connected by a complex transformation, and that they can be linearized to the equation $u''' = 0$ by a contact transformations. Equation (12) (H) is specifically linearized by the Legendre transformation. A set of real-valued linearizing transformations for equation (12) (C) was obtained in [9]. The linearizing transformations are complicated for constructing the general solution of these equations. Let us apply the novel algorithm to derive the general solution of the equation

$$v''' = k\frac{v'v''^2}{1+v'^2}, \tag{13}$$

where k is an arbitrary constant. One can check that conditions (11) are satisfied. Equation (10) becomes

$$y'' + y'^2\left(\frac{1}{y} - k\frac{y}{1+y^2}\right) = 0.$$

A linearizing transformation

$$t = \varphi(v, y), \qquad u = \psi(v, y)$$

is obtained by finding any solution of the involutive system

$$\psi_{vv} = \frac{\varphi_{vv}}{\varphi_v}\psi_v, \qquad \psi_{vy} = \frac{\varphi_{vv}}{2\varphi_v}\psi_y, \qquad \psi_{yy} = \left(\frac{1}{y} - k\frac{y}{1+y^2}\right)\psi_y, \qquad \varphi_{vvv} = \frac{3\varphi_{vv}^2}{2\varphi_v}.$$

For example, by choosing

$$\varphi(v, y) = v, \qquad \psi(v, y) = v + h(y)$$

one only needs to solve the equation

$$h' = C\frac{y}{(1+y^2)^{k/2}},$$

where C is an arbitrary constant.

If $k = 2$, then

$$h = C\ln(1+y^2),$$

and, hence, the general solution of equation (13) is obtained by taking the quadrature

$$\int (C_1 e^{C_2 v} - 1)^{-1/2}\,\mathrm{d}v = \pm x + C_3,$$

where C_i are arbitrary constants.

If $k \neq 2$, then

$$h = C(1+y^2)^{1-k/2},$$

and, hence, the general solution of equation (13) is obtained by taking the quadrature

$$\int \left((C_1 v + C_2)^{2/(2-k)} - 1\right)\mathrm{d}v = \pm x + C_3.$$

Let us also consider the equation linearized by GLT [20]:

$$y_{xxx} - \frac{y_x y_{xx}}{y} - 4\alpha y^2 y_x = 0, \quad (\alpha = \text{const}). \tag{14}$$

This equation is given in [20] as an equation which cannot be linearized by point, contact or generalized Sundman transformations. Direct calculations show that equation (14) satisfies criteria (11). This means that equation (14) can be linearized by the novel algorithm.

A search of many particular solutions of equations studied in mathematical physics is reduced to solving ordinary differential equations of the form

$$y''' + g_1(y, y')y'' + g_0(y, y') = 0. \tag{15}$$

In this case the general solution of equations (11) is

$$g_1 = \alpha_1 y' + \alpha_2 y'^3, \qquad g_0 = \beta_1 y' + \beta_2 y'^3, \tag{16}$$

where $\alpha_i(y)$, $\beta_i(y)$, $(i = 1, 2)$ are arbitrary functions. Hence, any equation of the form (11), (13) is linearizable. For the particular case $\alpha_2 = 0$, the substitution[6] $v(y) = y'^2$ leads to the linear equation

$$v'' + \alpha_1 v' + 2(\beta_1 + v\beta_2) = 0. \tag{17}$$

3. Stationary solutions of equations of fluid with internal inertia

The idea of a new way for linearizing ordinary differential equations was obtained during the study [30] of stationary solutions of the equations [26]

$$\begin{gathered} \dot\rho + \rho\,\mathrm{div}(u) = 0, \qquad \rho\dot u + \nabla p = 0 \\ p = \rho\frac{\delta W}{\delta\rho} - W = \rho\left(\frac{\partial W}{\partial\rho} - \frac{\partial}{\partial t}\left(\frac{\partial W}{\partial\dot\rho}\right) - \mathrm{div}\left(\frac{\partial W}{\partial\dot\rho}u\right)\right) - W. \end{gathered} \tag{18}$$

Here t is time, ∇ is the gradient operator with respect to space variables, ρ is the fluid density, u is the velocity field, $W(\rho, \dot\rho)$ is a given potential, 'dot' denotes the material time derivative: $\dot f = \frac{\mathrm{d}f}{\mathrm{d}t} = f_t + u\nabla f$, $\frac{\delta W}{\delta\rho}$ denotes the variational derivative of W with respect to ρ at the fixed value of u.

Let us consider a stationary solution[7]

$$u = U(x), \qquad \rho = R(x).$$

Substitution into equations (18) gives $U = k/R$ and

$$\begin{aligned} -R'''W_{\dot\rho\dot\rho}R^3 &- (R'')^2 W_{\dot\rho\dot\rho\dot\rho}kR^2 + 2R''R'R(R'W_{\dot\rho\dot\rho\dot\rho}k + 2W_{\dot\rho\dot\rho}R - W_{\rho\dot\rho\dot\rho}R^2) \\ &- (R')^4 W_{\dot\rho\dot\rho\dot\rho}k + (R')^3 R(2W_{\rho\dot\rho\dot\rho}R - 3W_{\dot\rho\dot\rho}) - (R')^2 W_{\rho\rho\dot\rho}k^{-1}R^4 \\ &+ R'R^2(W_{\rho\rho}k^{-2}R^3 - 1) = 0, \end{aligned} \tag{19}$$

where the function $W(\rho, \dot\rho) = W(R, kR'/R)$. Assuming that $R' = f(R)$, the last equation is reduced to a second-order ordinary differential equation. Applying the Lie test, one finds that the reduced equation can be linearized. For example, let

$$W = -a\rho^{-3}\dot\rho^2 + \beta\rho^3$$

with some constants a and β. Equation (19) becomes

$$2ak^2(R'''R^2 - 10R''R'R + 15(R')^3) + R'R^4(6\beta R^4 - k^2) = 0, \tag{20}$$

[6] It was noted by one of the referees.

[7] Similar result is obtained for a travelling wave.

and the second-order ordinary differential equation for the function $f(R)$ is

$$f'' + \frac{1}{f}(f')^2 - \frac{10}{R}f' + \frac{15f}{R^2} + \frac{R^2}{f}(q_1R^4 - q_2) = 0, \tag{21}$$

where $q_1 = \beta/(3ak^2)$, $q_2 = (78a)^{-1}$. This equation has the form (15) and (16), and it is transformed to the free particle equation $z''(\tau) = 0$ by the change

$$z = R^{-1}f^2(R) + q_1R^{-9} - q_2R^{-13}, \qquad \tau = R^{-7}.$$

Hence, the general solution of equation (21) is

$$f^2(R) = R^{-12}(C_1R^{13} + C_2R^6 + q_1R^4 - q_2),$$

where C_1 and C_2 are arbitrary constants. The function $R(x)$ is found by the quadrature $R' = f(R)$.

4. Linearization of travelling waves of PDE

Solutions of many partial differential equations were obtained by assuming that a solution is a travelling wave type.

4.1. The Korteveg–de Vries (KdV) equation

The Korteveg–de Vries equation

$$u_t + uu_x + Ku_{xxx} = 0, \qquad (K > 0)$$

has attracted enormous attention over the years and has served as the model equation for the development of soliton theory. Of particular interest among the solutions of the KdV equation are travelling wave solutions:

$$u(x,t) = H(x + Dt).$$

Substituting the representation of a solution into the KdV equation, one finds

$$KH''' + (H + D)H' = 0. \tag{22}$$

This equation has the form (15) and (16); hence, equation (22) is linearizable by the algorithm developed in section 2.

Note that equation (22) cannot be linearized by point, contact or generalized Sundman transformations.

It is worth noting that equation which is obtained after integrating (22),

$$KH'' + H^2/2 + DH = C,$$

does not satisfy the Lie test (3).

4.2. One class of third-order partial differential equations

Let us consider the nonlinear third-order partial differential equation

$$u_t + 2ku_x - \varepsilon u_{xxt} = \gamma uu_{xxx} + \alpha uu_x + \beta u_xu_{xx} - \Gamma u_{xxx}. \tag{23}$$

The following special cases of equation (23) have appeared in the literature[8]:

(a) the Fornberg–Whitham equation ($\varepsilon = 1, \alpha = -1, \beta = 3, \gamma = 1, \Gamma = 0$ and $k = 1/2$),
(b) the Rosenau–Hyman equation ($\varepsilon = 0, \alpha = 1, \beta = 3, \gamma = 1, \Gamma = 0$ and $k = 0$),

[8] Symmetries of equation (23) with $\Gamma = 0$ were studied in [31].

(c) the Fuchssteiner–Fokas–Camassa–Holm equation ($\varepsilon = 1, \alpha = -3, \beta = 2, \Gamma = 0$ and $\gamma = 1$),
(d) the Benjamin–Bona–Mahoney equation ($\varepsilon = 1, \alpha = 1, \beta = 0, \gamma = 0, \Gamma = 0$ and $k = 1/2$).
(e) the b-equation [32] ($\varepsilon = \mu^2, \alpha = 1 + b, \beta = b\mu^2, \gamma = -\mu^2$ and $k = -c_0/2$). The cases $\alpha = 3$ and $\alpha = 4$ exhaust the integrable candidates for the b-equation [32].

Substituting the travelling wave representation of a solution into equation (23), one gets

$$(\gamma H - \Gamma + \varepsilon D)H''' + (\alpha H - D - 2k)H' + \beta H'H'' = 0.$$

Here it is assumed that $(\varepsilon D - \Gamma)^2 + \gamma^2 \neq 0$. In order to include in the consideration the equation [19]

$$(\beta - 2)HH''' + \beta H'H'' - (\beta - 2)f(H)H' = 0, \tag{24}$$

where

$$f = q_2H^2 + q_0,$$

let us study a little bit more general equation

$$(\gamma H - \Gamma + \varepsilon D)H''' + (q_2H^2 + \alpha H + q_0 - D - 2k)H' + \beta H'H'' = 0. \tag{25}$$

Equation (24) is a generalization of the Euler equations (Euler top) describing a motion of a rigid body with the fixed centre of mass and corresponding moments of inertia

$$\dot{x} = yz, \qquad \dot{y} = zx, \qquad \dot{z} = xy.$$

Equation (25) has the form (15), (16), hence, it is linearizable.
Equation (25) can also be linearized by a point transformation if and only if

$$\beta(\beta - 3\gamma) = 0, \qquad D(\alpha\varepsilon + \gamma) + (2k + q_0)\gamma - \Gamma\alpha = 0,$$
$$\gamma q_2 = 0, \qquad (D\varepsilon - \Gamma)q_2 = 0.$$

The same conditions are necessary and sufficient for linearization (25) by contact or generalized Sundman transformations.

4.3. *One class of fourth-order partial differential equations*

Let us consider the nonlinear fourth-order partial differential equation [27]

$$(ku + \gamma u^2)_{xx} + \nu uu_{xxxx} + \mu u_{xxtt} + \alpha u_x u_{xxx} + \beta u_{xx}^2 - u_{tt} = 0,$$

where $\alpha, \beta, \gamma, \mu, \nu$ and k are arbitrary constants. If $\beta = \alpha - \nu$, then this equation can be written in the divergent form

$$(ku + \gamma u^2 + \nu uu_{xx} + \mu u_{tt})_{xx} + (\alpha - 2\nu)(u_x u_{xx})_x - u_{tt} = 0. \tag{26}$$

Integrating equation for a travelling wave of equation (26), one obtains

$$(\nu H + D^2\mu)H''' + (\alpha - \nu)H'H'' + (2\gamma H + k - D^2)H' = C, \tag{27}$$

where C is a constant of the integration. It is assumed that $\nu^2 + D^2\mu^2 \neq 0$. Since equation (27) has the form (15), it is linearizable if $C = 0$.

Equation (27) is linearizable by either point or contact or generalized Sundman transformation if and only if

$$(\alpha - \nu)(\alpha - 4\nu) = 0, \qquad C(\alpha - 4\nu) = 0, \qquad k\nu = D^2(2\gamma\mu + \nu).$$

4.4. The Boussinesq equation

The study of travelling waves of the Boussinesq equation

$$u_{tt} = \left(u_{xx} + \tfrac{1}{2}u^2\right)_{xx}$$

after one integration leads to the equation

$$H''' + (H - D^2)H' = C, \tag{28}$$

where C is a constant of the integration. This equation cannot be linearized by either point or contact or generalized Sundman transformation. Since equation (28) has the form (15), it is linearizable if and only if $C = 0$.

Note that after one more integration, equation (28) becomes

$$H'' + \tfrac{1}{2}H^2 - D^2H = C\xi + C_0. \tag{29}$$

Equation (29) fails the Lie test.

Remark. Similar statement is fair for travelling waves of the Kadomtsev–Petviashvili equation.

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